How would you solve the following equations?
The options are

a. integrate $y' =$ function of x
b. separate the variables
c. first order linear equation
d. make a special substitution
e. homogeneous equation
f. Bernoulli equation
g. exact equation
h. reduce the order

1. $xy' = y + \frac{y^2}{x}$ e or f

$v = \frac{y}{x}$, $x\frac{dv}{dx} = v^2$ separate the variables

2. $x\frac{dy}{dx} + 6y = 3xy^{4/3}$ f

$v = y^{1-4/3} = y^{-\frac{1}{3}}$

$\frac{dv}{dx} - \frac{2v}{x} = -1$, IF $e^{\int -\frac{2}{x}}$.

3. $y' = \sqrt{x+y+1}$ d

$v = \sqrt{x+y+1}$ ... $\frac{dv}{dx} = \frac{v+1}{2v}$ separate variables

4. $yy'' = (y')^2$ h

$v = y'$ ... $yv\frac{dv}{dy} = v^2$ separate.

5. $xy'' + y' = 4x$ c

IF $e^{\int \frac{1}{x}dx}$

6. $2xy\frac{dy}{dx} + y^2 = 10x$ g

$\frac{d}{dx}(xy^2) = 10x$

7. $xy^2 + 3y^2 - x^2y' = 0$ b

8. $2xy + x^2y' = y^2$ e

9. $x^3 + 3y - xy' = 0$ IF $e^{\int -\frac{3}{x}dx}$ c

- Calculate e^{At} where $A = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

(Note that A has eigenvalues 1,3 with eigenvectors $\begin{bmatrix}1\\ -1\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix}$

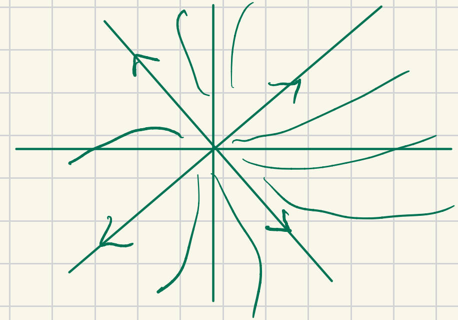
Also $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

a. $\begin{bmatrix}e^t & e^{3t}\\ -e^t & e^{3t}\end{bmatrix}$

b. $\begin{bmatrix}e^t & te^{3t}\\ -e^t & e^{3t}\end{bmatrix}$

c. $\frac{1}{2}\begin{bmatrix}e^t+e^{3t} & -e^t+e^{3t}\\ -e^t+3t & e^t+e^{3t}\end{bmatrix}$

- Classify all the critical points of x′ = Ax (as node (proper, improper), center, saddle point, spiral point, stable, unstable etc etc).



- How would you find a particular solution to x′ = Ax + $\begin{bmatrix}e^t\\ 0\end{bmatrix}$

See next page.

- How would you find a particular solution to $x' = Ax + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$ where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

a. Try a solution $\begin{bmatrix} a \\ b \end{bmatrix} e^t + \begin{bmatrix} c \\ d \end{bmatrix} e^{3t}$

b. Try a solution $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

c. Try a solution $\begin{bmatrix} a \\ b \end{bmatrix} e^t + \begin{bmatrix} c \\ d \end{bmatrix} t e^t$

d. something else.

How would you calculate e^{At} where

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

a. substitute At into the power series for e^u

b. solve the differential equation x′ = Ax and do a calculation with the fundamental matrix

c. break apart A into two pieces, and substitute those into the power series for e^u

d. something else.

Same question when $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$

Same question when $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$

Section 5 page 325 question 26 +

Which of the following is the appropriate form of a solution of the non-homogeneous equation

x'' - 6x' + 13x = t e^{3t} sin(2t)

Note that the characteristic polynomial has roots 3+2i and 3-2i

a. $ate^{3t}\sin 2t + bte^{3t}\cos 2t$

b. $ate^{3t}\sin 2t + bte^{3t}\cos 2t + ce^{3t}\sin 2t + de^{3t}\cos 2t$

c. $at^2e^{3t}\sin 2t + bt^2e^{3t}\cos 2t$

d. None of the above.

e. formula a + formula c.

What about x'' - 6x' + 13x = t sin(2t)

a. $ate^{3t}\sin 2t + bte^{3t}\cos 2t$

b. $at\sin 2t + bt\cos 2t + c\sin 2t + d\cos 2t$

Exam in Murphy 130

For things you don't need to know, only go by what I have told you you don't need to know.

Please check the grading if you got a low score on any question (in Exam 3)

Linear algebra

True/false for a linear equation $Ax = b$ where A is a 2 x 2 matrix?

- If A is invertible, then the above always has a unique solution for any b.
- If the rank of A is 1, we can always find a vector b in R^2 so that $Ax = b$ does not have a solution.
- Suppose $b = 0$. If the rank of A is 2, we can always find a solution $x \neq 0$
- Suppose v_1 and v_2 are both solutions. Then 3v_1 + 2v_2 is also a solution.
- If the column vectors of A are linearly independent, then A has an inverse.
- The 2 x 2 zero matrix is the only 2 x 2 matrix whose null space is 2-dimensional.

Questions not about a particular equation $Ax = b$. True/False ?

True at least one free variable False

- We can find a 2 x 3 matrix whose null space is the zero vector space. ✓ $0 + \leq 2 \neq 3$
- We can find a 2 x 3 matrix whose null space is the space spanned by the vector (1,2). ✓ Null space vectors have length 3
- ✓ We can find a 2 x 3 matrix whose null space is the space spanned by the vector (1,2,0). $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has the correct nullspace.
- We can find a 2 x 3 matrix whose null space is the space spanned by the vector (1,2,0) and whose column space is the space spanned by (2,1). ✓ $\underbrace{\text{dim nullsp}}_{1} + \underbrace{\text{rank}}_{1} \neq 3$
- ✓ We can find a 2 x 3 matrix whose null space is the space spanned by the vector (1,2,0) and whose column space has dimension 2.
- We can find a 2 x 3 matrix whose column space has dimension 3. ✓ rank ≤ 2

True/false

Let A be an $m \times n$ matrix with $m < n$.

1. $Ax = b$ always has a solution.
2. $Ax = 0$ always has a solution
3. $Ax = 0$ always has infinitely many solutions.
4. $Ax = 0$ always has a non-zero solution.
5. The columns of A are necessarily dependent.
6. The rows of A are necessarily independent.

Suppose now just that $m \leq n$

1. If $Ax = b$ has a solution for every b then the columns of A are independent.
2. If $Ax = b$ has a solution for every b then the columns of A span R^m
3. If $Ax = b$ has a solution for every b then the rank of A is n.

Page 107 question 7.
A car starts from rest and its engine accelerates it at 10 ft/sec^2
Also, air resistance provides 0.1 ft/sec^2 of deceleration for each ft/sec of the car's velocity.
(a) Find the car's maximum possible velocity.
(b) How long does it take to attain 90% of the limiting velocity, and how far does it travel in doing this?

Start with:
What equation is relevant for this problem?

Let $x(t)$ = position at time t,

a. $x'' + 10x' + \frac{1}{10} = 0$ $\quad x'' = 10 - 0.1x'$

✓ b. $x'' + \frac{x'}{10} - 10 = 0$

✓ c. $\frac{dv}{dt} + \frac{v}{10} - 10 = 0$ $\quad v = x'$

d. None of the above

Question (a): Solve $x'' = 0$

a. 10 b. 90 c. 100 ✓ d. 110 e. 1000

Solve $\frac{dv}{dt} + \frac{v}{10} = 10$

Separate variables, etc.

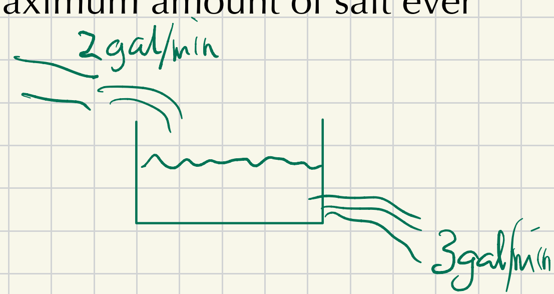
Get $v = \cdots$. Solve $v = 90$

to get t.

How far? $x = \int v\,dt$

and put in t.

Page 54 question 36.
A tank contains 60 gallons of pure water. Brine with concentration 1 lb salt per gallon enters at 2 gal/min. Perfectly mixed solution leaves at 3 gal/min. Thus the tank is empty after 1 hour.
(a) Find the amount of salt in the tank after t minutes,
(b) What is the maximum amount of salt ever in the tank?



What is the volume of liquid in the tank at time t?

a $t+60$ b $60t$ c $60-t$ ✓

d None of the above

What equation is appropriate for solving this problem?

✓a $\frac{dx}{dt} = 2 - \frac{3x}{60-t}$

b $\frac{dx}{dt} = \frac{2}{60-t} - 3x$

c $\frac{dx}{dt} = \frac{2-3x}{60-t}$

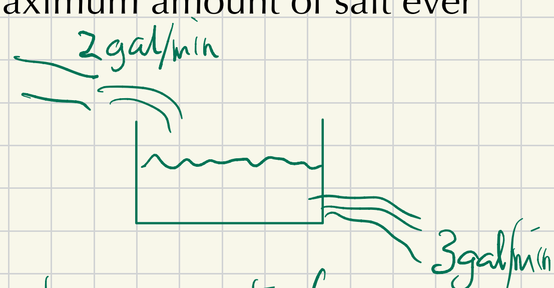
d None of the above.

Let $x(t)$ lb salt be in the tank. $\frac{x}{60-t}$ is the concentration

How do we go about solving this equation? Integrating factor etc.

Page 54 question 36.
A tank contains 60 gallons of pure water. Brine with concentration 1 lb salt per gallon enters at 2 gal/min. Perfectly mixed solution leaves at 3 gal/min. Thus the tank is empty after 1 hour.
(a) Find the amount of salt in the tank after t minutes,
(b) What is the maximum amount of salt ever in the tank?

Solution :



Let $x(t)$ be the amount of salt in the tank at time t.

The volume of liquid in the tank at time t is $60 - t$

The concentration of salt at time t is $\frac{x(t)}{60-t}$ lb/gal

We get $\frac{dx}{dt} = 2 - \frac{3x}{60-t}$

$$\frac{dx}{dt} + \frac{3x}{60-t} = 2$$

$$\text{I.F.} = e^{\int \frac{3}{60-t}dt} = e^{-3\ln(60-t)}$$

$$= e^{\ln(60-t)^{-3}} = \frac{1}{(60-t)^3}$$

$$\frac{1}{(60-t)^3}\frac{dx}{dt} + \frac{3x}{(60-t)^4} = \frac{2}{(60-t)^3}$$

$$\frac{d}{dt}\frac{x}{(60-t)^3} = \frac{2}{(60-t)^3}$$

$$\frac{d}{dt}\frac{x}{(60-t)^3} = \frac{2}{(60-t)^3}$$

$$\frac{x}{(60-t)^3} = \frac{1}{(60-t)^2} + C$$

$$x = 60 - t + C(60-t)^3$$

$$x(0) = 0 = 60 + C\,60^3$$

$$C = -\frac{1}{60^2}$$

$$x = 60 - t - \frac{(60-t)^3}{60^2}$$

To find the maximum solve

$$\frac{dx}{dt} = 0$$

Like section 3.2 questions 23-26:

Determine for what values of k and c the system has (a) a unique solution (b) no solution (c) infinitely many solutions

$$3x + 2y = 1$$
$$6x + cy = k \quad ,$$

Like section 9.1 13-20

Identity whether the critical point (0,0) is stable, asymptotically stable ur unstable. From aspects of the solutions, identify it as a node, a saddle point, a center or a spiral point.

20. dx/dt = y, dy/dt = -5x - 4y

$\lambda = -2 \pm i$ Asymptotically Stable spiral point.

This gives solutions $e^{-2t}\begin{bmatrix} \cos t \cdots \\ \cdots \end{bmatrix}$ $e^{-2t}\begin{bmatrix} \cos t \\ \vdots \end{bmatrix}$

We do not need to distinguish proper and improper nodes.